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Productionstructures and external diseconomies

Research memorandum

TILBURG INSTITUTE OF ECONOMICS
DEPARTMENT OF ECONOMICS

Productionstructures and External diseconomies

by

W.M. van den Goorbergh

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PART I

Introduction

A modern and systematic treatment of the theory of cost and production functions was given by Shephard [4]. In the present paper an attempt is made to integrate the Shephardian productionstructures and the external diseconomies, that production can cause on consumption.
In part 2 of this paper the conditions for integration are studied and the construction of productionstructures with properties similar to that of Shephard's is carried out.
In part 3 the possibility of restricting the level of external diseconomies by pricing is paid attention to. Some remarks will be made on the interpretation of the model in a macro- as well as in a micro-economic sense. On this point the introduction of the concept of a social utility function is inevitable.

PART 2

Production theory

2.1 Introduction

The reader is assumed to be familiar with the concept of cost and production theory develloped by Shephard [4]. As a reminder the definitions and properties that play a major role in this paper are stated below:

(2.1.1) Definition:

A production inputset $L(u)$ of a technology is the set of all input vectors x yielding at least the outputrate u, for $u \in [0,+\infty)$.

(2.1.2) Definition:

The efficient subset $E(u)$ of a production inputset $L(u)$ is given by $E(u) = \{x \mid x \in L(u),\ y \leq x \Rightarrow y \notin L(u)\}$.

(2.1.3) Definition:

A production technology is a family of inputsets T: $L(u)$, $u \in [0,+\infty)$ satisfying:

P.1 $\quad L(0) = D,\ 0 \notin L(u) \quad \text{for } u > 0$ [1)]

P.2 $\quad x \in L(u) \wedge x^1 \geqq x \quad \Rightarrow \quad x^1 \in L(u)$

P.3 $$(x>0) \vee [\,(x \geq 0) \wedge \{\exists_{\bar{\lambda}>0}\ \exists_{\bar{u}>0}\ (\bar{\lambda}x) \in L(\bar{u})\}] \Rightarrow$$
$$\Rightarrow \forall_{u \geq 0}\ \exists_{\lambda \geq 0}\ \lambda x \in L(u)$$

P.4 $\quad u_2 \geq u_1 \geq 0 \quad \Rightarrow \quad L(u_2) \subset L(u_1)$

P.5 $\quad \bigcap_{0 \leq u \leq u_o} L(u) = L(u_o) \quad \text{for } u_o > 0$

---

1) $\quad D = \{x \mid x \geqq 0,\ x \in R^n\}$

P.6 $\bigcap_{u \geq 0} L(u) = \emptyset$

P.7 $\forall_{u \geq 0}$ $L(u)$ is closed

P.8 $\forall_{u \geq 0}$ $L(u)$ is convex

P.9 $\forall_{u \geq 0}$ $E(u)$ is bounded

(2.1.4) <u>Proposition</u>:

The production function $\Phi(x) = \text{Max}\,\{u \mid x \in L(u),\ u \geq 0\}$, $x \in D$, defined on the inputsets $L(u)$ of a technology with the properties P.1,...,P.9, has the following properties:

A.1 $\Phi(0) = 0$

A.2 $\forall_{x \in D}$: $\Phi(x)$ is finite

A.3 $x^1 \geqq x \quad \Rightarrow \quad \Phi(x^1) \geq \Phi(x)$

A.4 $(\forall_{x>0}) \vee \{\forall_{x \geq 0} \; \exists_{\lambda > 0} \; \Phi(\lambda x) > 0\} : \lim_{\lambda \to +\infty} \Phi(\lambda x) = +\infty$

A.5 $\Phi(x)$ is upper semi-continuous on D. [1]

A.6 $\Phi(x)$ is quasi-concave on D. [2]

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[1] The function $\Phi(x)$ is upper semi-continuous at a point x^0, if and only if for all $\varepsilon > 0$ there exists a neighbourhood $N_\varepsilon(x^0)$ of n^0 such that $x \in N_\varepsilon(x^0)$ implies $\Phi(x) < \Phi(x^0) + \varepsilon$.

[2] A numerical function $\Phi(x)$ defined on a convex subset $D \subset R^n$ is quasi-concave on D if for all points x and y of D.
$\Phi\{(1-\Theta)x+\Theta y\} \geq \text{Min}\,[\Phi(x), \Phi(y)]$ for all $\Theta \in [0,1]$

2.2 External diseconomies

It is well known that external diseconomies can occur during the production of a desired commodity. In a technological sense an external diseconomy is an (adjoining-) output of a productionprocess. Economically an external diseconomy should rather be interpreted as use of relatively scarce means and hence be considered as input of a productionprocess.
Now conditions will be formulated, for which external diseconomies can be treated as inputs in a production-structure à la Shephard. Then a "production function" is introduced, by which the external diseconomies can be eliminated. The properties of this function will be such as to enable us to construct a new Shephardian production-structure with external diseconomies treated as inputs. Finally a production technology will be considered for which the external diseconomies are bounded by an upper-limit.

2.3 The e.d.g.-function

A relation is assumed between the level of the external diseconomy and the outputrate u of the productionprocess. There may be several diseconomies involved, so for each external diseconomy i (i = 1...m) an external diseconomy generating (e.d.g.) function is defined with u as the independent variable. The properties of these functions $f_i(u)$ are assumed to be:

f.1 $\forall_i : f_i(0) = 0$

f.2 $\forall_i : u > 0 \quad \Rightarrow \quad f_i(u) > 0$

f.3 $\forall_i : u^1 \geq u \quad \Rightarrow \quad f_i(u^1) \geq f_i(u)$

f.4 $\forall_i : f_i(u)$ is finite for finite u

f.5 $\forall_i :$ if $u \to \infty$, then $f_i(u) \to \infty$

Clearly these properties are not highly restrictive, so the choisespace for specification of the e.d.g.-function is relatively large.

2.4 The construction of $\bar{L}(u)$

Summing up the levels of the external diseconomies in the vector $z = (z_1 \dots z_m)$ and the levels of the e.d.g.-functions in the vector $F(u) = \{f_1(u), \dots f_m(u)\}$ the following definition can be stated:

(2.4.1) Definition:

A vector (x,z) belongs to the productiontechnology $\bar{L}(u)$, if x belongs to a technology $L(u)$ with properties P1,...P9 (2.1.3) and if $z \geq F(u)$, so:

$$\bar{L}(u) = \{(x,z) \mid x \in L(u),\ z \geqq F(u)\},\ (x,z) \in D_{n+m}$$

Now the proof is given, that $\bar{L}(u)$ is satisfying the properties $\bar{P}.1 \dots \bar{P}.9$ of a Shephardian technology.

$$\bar{P}.1 \quad \left.\begin{array}{l} \text{For } u = 0;\ \forall_{x \in D_n} : x \in L(u) \text{ See P.1} \\ \qquad ;\ F(u) = 0 \ (f1) \Rightarrow \forall_{z \in D_m} : z \geqq F(u) \end{array}\right\} \Rightarrow \bar{L}(0) = D_{n+m}$$

$$\left.\begin{array}{l} \text{For } u > 0;\ 0 \notin L(u) \quad \text{See P.1} \Rightarrow \ x \geq 0 \\ \qquad ;\ F(u) > 0 \quad (f2) \qquad \Rightarrow \ z > 0 \end{array}\right\} \Rightarrow (x,z) \neq 0 \ \therefore 0 \notin \bar{L}(u)$$

$$\bar{P}.2 \quad \left.\begin{array}{l} \left.\begin{array}{l} (x,z) \in \bar{L}(u) \Rightarrow x \in L(u) \\ (x^1,z^1) \geqq (x,z) \Rightarrow x^1 \geqq x \end{array}\right\} \Rightarrow x^1 \in L(u) \text{ See P.2} \\ \left.\begin{array}{l} (x,z) \in \bar{L}(u) \Rightarrow z \geqq F(u) \\ (x^1,z^1) \geqq (x,z) \Rightarrow z^1 \geqq z \end{array}\right\} \Rightarrow z^1 \geqq F(u) \end{array}\right\} \Rightarrow (x^1,z^1) \in \bar{L}(u)$$

$\bar{P}$.3 For $(x,z) > 0$; $x > 0$ so $\forall_{u \geq 0} \; \exists_{\lambda_1 \geq 0} : \lambda_1 x \in L(u)$. See P.3

; $z > 0$ so $\forall_{u \geq 0} \; \exists_{\lambda_2 \geq 0} : \lambda_2 z \geqq F(u)$.

Let $\lambda_0 = \text{Max}\,[\lambda_1 \lambda_2]$ so:

$\forall_{u \geq 0} \; \exists_{\lambda_0 \geq 0} : \lambda_0 x \in L(u)$. See P.2

$\forall_{u \geq 0} \; \exists_{\lambda_0 \geq 0} : \lambda_0 z \geqq F(u)$.

Consequently for $(x,z) > 0$:

$\forall_{u \geq 0} \; \exists_{\lambda_0 \geq 0} : \lambda_0 (x,z) \in \bar{L}(u)$

For $(x,z) \geq 0$ three cases should be distinguished:

a. $x = 0$ and $z \geq 0$. This case can be ignored for $x = 0 \notin L(u)$ if $u > 0$. See P.1

b. $x \geq 0$ and $\exists_i : z_i = 0$. This case can be ignored too, for $z_i = 0 \ngeq f_i(u)$ if $u > 0$. See (f2)

c. $x \geq 0$ and $z > 0$

if $x \geq 0$ and $\exists_{\bar{\lambda} > 0} \; \exists_{\bar{u} > 0} : \bar{\lambda} x \in L(\bar{u})$

then $\forall_{u \geq 0,} \; \exists_{\lambda_1 \geq 0} : \lambda_1 x \in L(u)$. P.3

if $z > 0$ then $\forall_{u \geq 0,} \; \exists_{\lambda_2 \geq 0} : \lambda_2 z \geqq F(u)$

Let $\lambda_0 = \text{Max}\,[\lambda_1 \lambda_2]$ so $\forall_{u \geq 0,} \; \exists_{\lambda_0 \geq 0} : \lambda_0 x \in L(u)$. P.2

$\forall_{u \geq 0,} \; \exists_{\lambda_0 \geq 0} : \lambda_0 z \geqq F(u)$

Consequently for $(x,z) \geq 0$ holds:

If $\underset{\bar{\lambda}>0}{\exists}\ \underset{\bar{u}>0}{\exists} : \bar{\lambda}(x,z) \in \bar{L}(\bar{u})$, then $\underset{u\geq 0}{\forall}\ \underset{\lambda_0\geq 0}{\exists}\ \lambda_0(x,z) \in \bar{L}(u)$

$\bar{P}$.4 For $u_2 \geq u_1 \geq 0$;

$$\left.\begin{array}{l} \text{If } x \in L(u_2) \text{ then } x \in L(u_1). \text{ See P.4} \\ ;\ F(u_2) \geqq F(u_1). \text{ See (f3)} \Rightarrow \text{ If } z \geqq F(u_2), \text{ then } z \geqq F(u_1) \end{array}\right\} \Rightarrow$$

$\Rightarrow$ If $(x,z) \in \bar{L}(u_2)$, then $(x,z) \in \bar{L}(u_1)$.

Consequently $\bar{L}(u_2) \subset \bar{L}(u_1)$

$$\bar{P}.5 \quad \left.\begin{array}{l} \bigcap_{0\leq u\leq u_o} L(u) = L(u_o). \text{ See P.5} \\ \bigcap_{0\leq u\leq u_o} \{z \mid z \geqq F(u)\} = \{z \mid z \geqq F(u_o)\} \text{ See (f3)} \end{array}\right\} \Rightarrow \bigcap_{0\leq u\leq u_o} \bar{L}(u) = \bar{L}(u_o)$$

$$\bar{P}.6 \quad \left.\begin{array}{l} \bigcap_{u\geq 0} L(u) = \emptyset. \text{ See P.6} \\ \bigcap_{u\geq 0} \{z \mid z \geqq F(u)\} = \{z \mid z \geqq \infty\} = \emptyset \\ \qquad \text{Due to (f3) and (f5)} \end{array}\right\} \bigcap_{u\geq 0} \bar{L}(u) = \emptyset$$

$$\bar{P}.7 \quad \left.\begin{array}{l} \underset{u\geq 0}{\forall} : L(u) \text{ is closed. See P.7} \\ \underset{u\geq 0}{\forall} : \{z \mid z \geqq F(u)\} \text{ is closed} \end{array}\right\} \Rightarrow \underset{u\geq 0}{\forall} : \bar{L}(u) \text{ is closed}$$

$$\bar{P}.8 \quad \left.\begin{array}{l} \left.\begin{array}{l} (x,z) \in \bar{L}(u) \Rightarrow x \in L(u) \\ (y,w) \in \bar{L}(u) \Rightarrow y \in L(u) \end{array}\right\} \Rightarrow \underset{\lambda \in [0,1]}{\forall} : \lambda x + (1-\lambda) y \in L(u) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{See P.8} \\ \left.\begin{array}{l} (x,z) \in \bar{L}(u) \Rightarrow z \geqq F(u) \\ (y,w) \in \bar{L}(u) \Rightarrow w \geqq F(u) \end{array}\right\} \Rightarrow \underset{\lambda \in [0,1]}{\forall} : \lambda z + (1-\lambda) w \geqq F(u) \end{array}\right\} \Rightarrow$$

$\Rightarrow \forall_{\lambda \in [0,1]} : \lambda(x,z) + (1-\lambda)\ (y,w) \in \bar{L}(u)$

Hence: $\forall_{u \geq 0} : \bar{L}(u)$ is convex.

(2.4.2) Definition:

The efficient subset $\bar{E}(u)$ of $\bar{L}(u)$ is given by

$\bar{E}(u) = \{(x,z) \mid (x,z) \in \bar{L}(u) \wedge (y,w) \leq (x,z) \Rightarrow (y,w) \notin \bar{L}(u)\}$

Clearly:

$\bar{E}(u) = \{(x,z) \mid x \in E(u),\ z = F(u)\}$

$\bar{P}.9$ $\left.\begin{array}{l} \forall_{u \geq 0} : E(u) \text{ is bounded. See P.9} \\ \{z \mid z = F(u)\} \text{ is also bounded by (f4)} \end{array}\right\} \Rightarrow \forall_{u \geq 0} : \bar{E}(u)$ is bounded

2.5 The e.d.e.-function

It is assumed that an external diseconomy can be eliminated partially or entirely by employing some combination of productionfactors. Hence for each external diseconomy i (i = 1...m) an external diseconomy eliminating (e.d.e.) -function is defined with x as independent variable. The properties of these functions $g_i(x)$ are assumed to be:

g.1 $\forall_i : g_i(x) = 0$ for $x \leqq 0$

g.2 $\forall_i : g_i(x)$ is finite for finite x

g.3 $\forall_i : x^1 \geqq x \Rightarrow g_i(x^1) \geq g_i(x)$

g.4 $\forall_i :$ if $x \to \infty$ then $g_i(x) \to \infty$

g.5 $\forall_i : g_i(x)$ is concave on D. [1)]

---

1) A numerical function g(x) defined on a convex subset $D \subset R^n$ is concave on D if for all points x and y of D, $g\{(1-\Theta)x+\Theta y\} \geq (1-\Theta)g(x)+\Theta g(y)$ for all $\Theta \in [0,1]$.

The last one of these properties is very restrictive. It is introduced to prove the convexity of the sets to be constructed. Economically property 5 restricts the specification of the e.d.e.-function to the class of production-functions of non-increasing returns to scale.

2.6 The construction of $\bar{\bar{L}}(u)$

(2.6.1) Definition:

A vector (x,z) belongs to a productiontechnology $\bar{\bar{L}}(u)$, if there exists such a $(m+1)$-partition of x that x^0 belongs to a technology $L(u)$ with properties P.1...P.9 (2.1.3) and if for all i $(i = 1 \ldots m)$ holds:

$$z_i \geq f_i(u) - g_i(x^i) \geq 0, \text{ hence}$$

$$\bar{\bar{L}}(u) = \{(x,z) \mid x^0 \in L(u),\ z_i \geq f_i(u) - g_i(x^i),\ \sum_{i=0}^{m} x^i = x,\ x^i \geq 0\} \cap D_{n+m}$$

Analogously the reasoning in (2.4) it can be proved that $\bar{\bar{L}}(u)$ is satisfying the properties $\bar{\bar{P}}.1 \ldots \bar{\bar{P}}.9$ of a Shephardian technology. Only the property of convexity will be proved here:

$$\bar{\bar{P}}.8 \quad \left.\begin{array}{l}(x,z) \in \bar{\bar{L}}(u) \Rightarrow x^0 \in L(u) \\ (y,w) \in \bar{\bar{L}}(u) \Rightarrow y^0 \in L(u)\end{array}\right\} \Rightarrow \forall_{\lambda \in [0,1]} : \lambda x^0 + (1-\lambda) y^0 \in L(u)$$

See P.8 (a)

$$\left.\begin{array}{l}(x,z) \in \bar{\bar{L}}(u) \Rightarrow z_i \geq f_i(u) - g_i(x^i) \\ (y,w) \in \bar{\bar{L}}(u) \Rightarrow w_i \geq f_i(u) - g_i(y^i)\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l}\Rightarrow \forall_{\lambda \in [0,1]} : \lambda z_i + (1-\lambda) w_i \geq f_i(u) - [\lambda g_i(x^i) + (1-\lambda) g_i(y^i)] \\ \qquad \text{But due to (g.5)} \\ \forall_{\lambda \in [0,1]} : \lambda g_i(x^i) + (1-\lambda) g_i(y^i) \leq g_i\,[\lambda x^i + (1-\lambda) y^i]\end{array}\right\} \Rightarrow$$

$$\Rightarrow \underset{\lambda\in[0,1]}{\forall} : \lambda z_i+(1-\lambda)w_i \geq f_i(u) - g_i\,[\lambda x^i+(1-\lambda)y^i] \quad \text{(b)}$$

Moreover:

$$\left.\begin{array}{l}\sum_{i=0}^{m} x^i = x \\ \\ \sum_{i=0}^{m} y^i = y\end{array}\right\} \Rightarrow \sum_{i=0}^{m} [\lambda x^i+(1-\lambda)y^i] = \lambda x+(1-\lambda)y \quad \text{for } \lambda\in[0,1]$$

$$\left.\begin{array}{l}x^i \geq 0 \\ \\ y^i \geq 0\end{array}\right\} \Rightarrow \lambda x^i+(1-\lambda)y^i \geq 0 \quad \text{for} \quad \lambda\in[0,1]$$

$$\left.\begin{array}{l}\text{(a)} \\ \\ \text{(b)}\end{array}\right\} \Rightarrow \underset{\lambda\in[0,1]}{\forall} : \lambda(x,z)+(1-\lambda)(y,w) \in \bar{\bar{L}}(u)$$

2.7 The construction of $\overset{0}{L}(u,\bar{z})$

Now we are able to construct a technology, for which the external diseconomies are bounded by a certain upper-limit. Let $\bar{z} = (\bar{z},\ldots\bar{z}_m)$ be the vector, summing up the upper-limits of the external diseconomies.

(2.7.1) Definition:

1. $B\bar{z} = \{(x,z) \mid x \in D_n, z = \bar{z}\}$

2. $B(u,\bar{z}) = \bar{\bar{L}}(u) \cap B\bar{z}$

3. $C(u,\bar{z}) = B(u,\bar{z})-(0,\bar{z})$ i.e. $(x,z) \in B(u,\bar{z}) \Leftrightarrow x \in C(u,\bar{z})$

Both $B\bar{z}$ and $B(u,\bar{z})$ are defined in R^{n+m}, while $C(u,\bar{z})$ is defined in R^n by a minor transformation.

(2.7.2) Definition:

A vector x belongs to a productiontechnology $\overset{0}{L}(u,\bar{z})$ with bounded external diseconomies, if there exists such a (m+1)-partition of x, that x^0 belongs to a technology L(u) with properties P.1...P.9 (2.1.3) and if for all i (i = 1...m) holds:

$$\bar{z}_i \geq f_i(u) - g_i(x^i) \geq 0, \text{ hence:}$$

$$\overset{0}{L}(u,z) = \{x \mid x^0 \in L(u),\ \bar{z}_i \geq f_i(u) - g_i(x^i),\ \sum_{i=0}^{m} x^i = x,\ x^i \geq 0\}$$

$$= \{x \mid x \in C(u,\bar{z})\}$$

It is easy to see that $\overset{0}{L}(u,\bar{z}) \subset L(u)$. For if $x \in \overset{0}{L}(u,\bar{z})$, then $x^0 \in L(u)$ and as $x \geqq x^0$, so $x \in L(u)$ due to P.2. Along analogous lines it can be proved that $\overset{0}{L}(u,\bar{z})$ is satisfying the properties $\overset{0}{P}$.1...$\overset{0}{P}$.9 of a Shephardian technology.

2.8 Linear homogeneous productionfunctions:

A productionstructure is said to be linear homogeneous, if the productionfunction defined on it is linear homogeneous. We will state the conditions, for which the constructed technologies $\bar{L}(u)$, $\bar{\bar{L}}(u)$ and $\overset{0}{L}(u,\bar{z})$ are homogeneous of degree one. First we state:

(2.8.1) Proposition:

If a productionfunction $\Phi(x)$ with properties A.1...A.6 is positively linear homogeneous in x, for all $\lambda \geq 0$ holds: if x belongs to the productioninputset L(u), that is associated with $\Phi(x)$ i.e. $L(u) = \{x \mid \Phi(x) \geq u\}$, then λx belongs to $L(\lambda u)$.

Proof:

If $x \in L(u)$ then $\Phi(x) \geq u$. So for all $\lambda \geq 0$: $\lambda\Phi(x) \geq \lambda u$ $\Phi(x)$ is linear homogeneous, hence $\Phi(\lambda x) \geq \lambda u$ so $\lambda x \in L(\lambda u)$.

$\bar{L}(u)$ is linear homogeneous if $(x,z) \in \bar{L}(u)$ implies $\lambda(x,z) \in \bar{L}(\lambda u)$ for $\lambda \geq 0$.

$(x,z) \in \bar{L}(u) \Rightarrow x \in L(u)$. $\lambda\, x \in L(\lambda u)$ for $L(u)$ is linear homogeneous

$\Rightarrow z \geqq F(u)$. So $\lambda z \geqq \lambda\, F(u)$ for $\lambda \geq 0$.
Hence $\lambda z \geqq F(\lambda u)$ if for all $\lambda \geq 0$:
$\lambda\, F(u) \geqq F(\lambda u)$

Hence $\bar{L}(u)$ is linear homogeneous for all $u \geq 0$ if:

1. $\forall_{u \geq 0}$: $L(u)$ is linear homogeneous.

2. $\forall_{u \geq 0,}\ \forall_{\lambda \geq 0}$: $\lambda\, F(u) \geqq F(\lambda u)$ i.e. $\lambda\, F(u) = F(\lambda u)$.

Hence all the e.d.g.-functions should be positively linear homogeneous.

By the same way of reasoning one can state that $\bar{\bar{L}}(u)$ is linear homogeneous for all $u \geq 0$ if:

1. $\forall_{u \geq 0}$: $L(u)$ is linear homogeneous.

2. $\forall_{u \geq 0,}\ \forall_{\lambda \geq 0,}\ \forall_{i}$: $\lambda\, f_i(u) \geq f_i(\lambda u)$ $\therefore$ $\lambda\, f_i(u) = f_i(\lambda u)$

3. $\forall_{x^i \geq 0,}\ \forall_{\lambda \geq 0,}\ \forall_{i}$: $\lambda g_i(x^i) \leq g_i(\lambda x^i)$ $\therefore$ $\lambda\, g_i(x^i) = g_i(\lambda x^i)$

All e.d.g.- and e.d.e.-functions should be positively linear homogeneous.

Finally it will be proved, that no conditions can be stated to quarantee $\overset{0}{L}(u,\bar{z})$ to be linear homogeneous.

Let $x \in \overset{0}{L}(u,\bar{z})$ i.e. $\bar{z}_i \geq f_i(u) - g_i(x^i)$ for all $u \geq 0$.

$$\left.\begin{array}{l} \text{Consider } \lambda > 1 \text{ so } \quad \bar{z}_i \geq \frac{1}{\lambda}\, f_i(u) - \frac{1}{\lambda}\, g_i(x^i) \text{ for all } u \geq 0 \\ \overset{0}{L}(u,\bar{z}) \text{ to be linear homogeneous, it should hold:} \\ \qquad \bar{z}_i \geq f_i(\lambda u) - g_i(\lambda x^i) \end{array}\right\} \Rightarrow$$

$\Rightarrow \frac{1}{\lambda} f_i(u) = f_i(\lambda u)$ for all $u \geq 0$. In accordance to (f.3) and (f.5) these conditions cannot be satisfied. This completes the proof.

2.9 An alternative e.d.g.-function

By assuming the level of the external diseconomies being determined exclusively by the outputrate u of the productionprocess, we are neglecting the possible influence of the specific production method selected from the productionpossibilities on the level of the external diseconomies. It can well be imagined, that in a two-factor technology a more capital intensive productionmethod causes a higher level of external diseconomies than a more labour intensive productionmethod, both methods yielding the same outputrate u.
Now an alternative e.d.g.-function will be defined.

(2.9.1) Definition:
For each external diseconomy i (i = 1...m) of a productionprocess an e.d.g.-function $h_i(x)$ is defined, x being the (efficient) inputvector of the productionprocess, satisfying the following properties:

h.1 $\forall_i$: $h_i(0) = 0$

h.2 $\forall_i$: $x \geq 0 \Rightarrow h_i(x) > 0$

h.3 $\forall_i$: $x^1 \geqq x \Rightarrow h_i(x^1) \geqq h_i(x)$

h.4 $\forall_i$: $h_i(x)$ is finite for finite x

h.5 $\forall_i$: If $x_i \to \infty$, then $h_i(x) \to \infty$

h.6 $\forall_i$: $h_i(x)$ is convex on D. [1]

The properties h.1...h.5 correspond to the five properties of the orginal e.d.g.-function. The last one is introduced to prove to convexity of the sets to be constructed (cfr.2.4). Economically we are dealing with a case of non-decreasing returns to scale.

2.10 The construction of $\bar{L}_1(u)$

Summing up the levels of the alternative e.d.g.-function in the vector $H(x) = \{h_1(x), h_2(x)...h_m(x)\}$ we can state the following definition:

(2.10.1) Definition:
A vector (x,z) belongs to a productiontechnology $\bar{L}_1(u)$, if x belongs to a technology L(u) with properties P.1...P.9 (2.1.3) and if $z \geqq H(x^1)$, $x^1 \leqq x$ and $\Phi(x^1) \geq u$. Hence:

$$\begin{aligned}\bar{L}_1(u) &= \{(x,z) \mid x \geqq x^1,\ \Phi(x^1) \geq u,\ z \geqq H(x^1)\} \quad (x,z) \in D_{n+m} \\ &= \{(x,z) \mid x \geqq x\ ,\ x^1 \in L(u),\ z \geqq H(x^1)\} \\ &= \{(x,z) \mid x \geqq x^1,\ x^1 \in E(u),\ z \geqq H(x^1)\}\end{aligned}$$

Along similar lines of reasoning as before it can be proved that $\bar{L}_1(u)$ is satisfying the properties $\bar{P}_1$.1...$\bar{P}_1$.9 of a Shephardian technology. Only the property of convexity will be proved here:

$\bar{P}_1$.8

$$\left.\begin{array}{l}(x,z) \in \bar{L}_1(u) \Rightarrow x^1 \in L(u) \\ (y,w) \in \bar{L}_1(u) \Rightarrow y^1 \in L(u)\end{array}\right\} \Rightarrow \forall_{\lambda\in[0,1]} : \lambda x^1 + (1-\lambda)y^1 \in L(u) \quad \text{See P.8}$$

---

[1]) The function h(x) is convex on D if for all points x and y of D, $h\{(1-\Theta)x+\Theta y\} \leq (1-\Theta)\ h(x) + \Theta\ h(y)$ for all $\Theta \in [0,1]$.

$$\left.\begin{array}{l}(x,z) \in \bar{L}_1(u) \Rightarrow z \geqq H(x^1) \\ (y,w) \in \bar{L}_1(u) \Rightarrow w \geqq H(y^1)\end{array}\right\} \Rightarrow \forall_{\lambda\in[0,1]} : \lambda z+(1-\lambda)w \geqq \lambda H(x^1)+(1-\lambda) H(y^1)$$

But due to (h.6) $\qquad \forall_{\lambda\in[0,1]} : H\{\lambda x^1+(1-\lambda)y^1\} \leqq \lambda H(x^1)+(1-\lambda)H(y^1)$ $\Rightarrow$

$$\Rightarrow \forall_{\lambda\in[0,1]} : \lambda z+(1-\lambda)w \geqq H\{\lambda x^1+(1-\lambda)y^1\}$$

Moreover: $\left.\begin{array}{l}x^1 \leqq x \Rightarrow \lambda x^1 \leqq \lambda x \\ y^1 \leqq y \Rightarrow (1-\lambda)y^1 \leqq (1-\lambda)y\end{array}\right\} \Rightarrow \lambda x^1+(1-\lambda)y^1 \leqq \lambda x+(1-\lambda)y$

Hence: $\forall_{\lambda\in[0,1]} : \lambda(x,z) + (1-\lambda)(y,w) \in \bar{L}_1(u)$

Consequently: $\forall_{u\geq 0} : \bar{L}_1(u)$ is convex

2.11 <u>The construction of $\bar{\bar{L}}_1(u)$ and $\overset{0}{L}_1(u,\bar{z})$</u>

Evidently in the same way as constructing an alternative for $\bar{L}(u)$, an alternative definition for $\bar{\bar{L}}(u)$ and $\overset{0}{L}(u,\bar{z})$ can be formulated.

(2.11.1) <u>Definition</u>:

$$\bar{\bar{L}}_1(u) = \{(x,z) \mid y^0 \leqq x^0,\ y^0 \in L(u),\ z_i \geq h_i(y^0) - g_i(x^i),\ \sum_{i=0}^{m} x^i = x,\ x^i \geq 0\} \cap D_{n+m}$$

(2.11.2) <u>Definition</u>:

$$\overset{0}{L}_1(u,\bar{z}) = \{x \mid y^0 \leqq x^0,\ y^0 \in L(u),\ \bar{z}_i \geq h_i(y^0) - g_i(x^i),\ \sum_{i=0}^{m} x^i = x,\ x^i \geq 0\}$$

The properties P.1...P.9 of (2.1.3) hold for these technologies too. As far as the linear homogenity of $\bar{L}_1(u)$ is concerned: this property is satisfied if $(x,z) \in \bar{L}_1(u)$ implies $\lambda(x,z) \in \bar{L}_1(\lambda u)$ for $\lambda \geq 0$

$$
\begin{aligned}
(x,z) \in \bar{L}_1(u) \quad \Rightarrow \quad & x^1 \leqq x, \quad x^1 \in L(u) \\
& \lambda x^1 \leqq \lambda x, \ \lambda x^1 \in L(\lambda u) \text{ for } L(u) \text{ is linear homogeneous} \\
\Rightarrow \quad & z \geqq H(x^1) \\
& \lambda z \geqq \lambda H(x^1), \ \lambda \geq 0 \\
& \lambda z \geqq H(\lambda x^1) \text{ if for all } \lambda \geq 0\text{:} \\
& \qquad \lambda H(x^1) \geqq H(\lambda x^1)
\end{aligned}
$$

Hence $\bar{L}_1(u)$ is linear homogeneous for all $u \geq 0$ if:

1. $\forall_{u \geq 0}$: $L(u)$ is linear homogeneous

2. $\forall_{u \geq 0,} \ \forall_{\lambda \geq 0}$: $\lambda H(x^1) \geqq H(x^1)$ i.e. $\lambda H(x^1) = H(\lambda x^1)$.

   Hence all the alternative e.d.g.-function should be posivily linear homogeneous.

PART 3

Pricetheory

3.1 Introduction

Now we have studied the conditions for which external diseconomies can be treated as inputs in a production-technology, we are heading for the problem of integration in the dual productionstructure, the pricestructure. Considering external diseconomies as utilization of relatively scarce means, the question of their economic evaluation i.e. their pricing can't remain unanswered. In the productionstructures $\bar{\bar{L}}(u)$ and $\bar{\bar{L}}_1(u)$ the external diseconomies can be considered as inputs of a production-process that can be substituted partially or entirely for "common" productionfactors.
Clearly there is a factorminimal cost function for the productionstructures mentioned above (See [4] page 79). Hence the economic evaluation of external diseconomies - e.g. expressed in taxation on causing them - together with a given price-vector for the "common" production-factors, influence the choise of the optimal combination of inputs, "common" factors as well as external diseconomies, for yielding some outputrate of the production-process, provided the condition of cost minimizing behaviour.
It will be shown that given some outputrate of the desired commodity such a minimal taxation or price for the external diseconomies can be established, that provided the pricevector for "common" factors and the condition of cost minimizing behaviour, the level of external diseconomies in the optimal inputvector is not exceeding a given maximum.
Some remarks will be made on the relation between the minimal taxation and the outputrate of the production-

process, the maximum that should not be exceeded and the pricevector of the "common" factors.
Finally the interpretation of the model will be discussed. Then we will get in touch with welfare economies by introducing a social utility function.

3.2 A particular hyperplane

(3.2.1) Proposition:

If $C \subset D_{n+m}$ is a convex, closed set, whose elements are denoted as (x,z) for $x \in D_n$ and $z \in D_m$ and $\bar{p} \in D_n$ is a pricevector and $\bar{z} \in D_m$ is a maximumvector, there exist a taxationvector $q \in D_m$, an element $(\bar{x},\bar{z}) \in C$ and a scalar β in such a way that $\bar{p}\ \bar{x} + q\ \bar{z} = \beta$

and $\bar{p}\ x + q\ z \geq \beta$ for $(x,z) \in C$.

Proof:

Let $A = \{(x,z) \mid x \in D_n,\ z = \bar{z}\}$. A is convex and closed. Consider $A \cap C$. (See fig.1). $A \cap C$ is convex and closed; hence there exist a point $(\bar{x},\bar{z}) \in \text{Bnd}\ (A \cap C)$ and a scalar α in such a way that $\bar{p}\ \bar{x} = \alpha$

and $\bar{p}\ \bar{x} \geq \alpha$ for $(x,z) \in A \cap C$.

Let $B = \{(x,z) \mid \bar{p}\ x = \alpha,\ z = \bar{z}\}$. B is convex and not empty (fig.1). Let Int C be the set of interior points of C; Int C is convex. Clearly $B \cap \text{Int}\ C = \emptyset$.
Since B and Int C are two convex, disjunct, non-empty sets in R^{n+m}, there is according to the first separation-theorem of Berge [1] a hyperplane V, separating both sets. So there exist a $p \in D_n$, a $q \in D_m$ and a $\beta \in \text{Re}$ such that

$V = \{(x,z) \mid p\ x + q\ z = \beta\}$

$p\ x + q\ z \geq \beta$ for $(x,z) \in \text{Int}\ C$

$p\ x + q\ z \leq \beta$ for $(x,z) \in B$

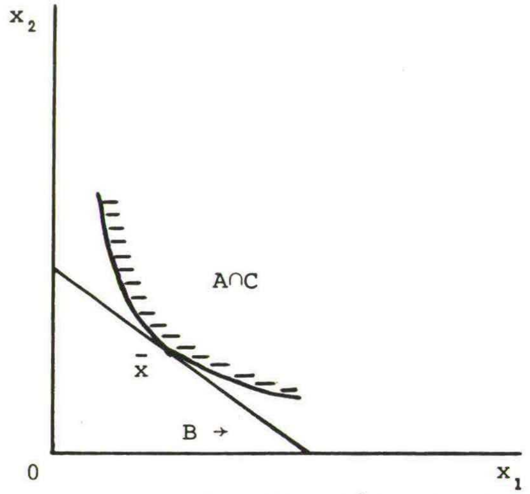

fig 1: $A \cap C = \{(x,z) \mid (x,z) \in C,\ z = \bar{z}\}$
$B = \{(x,z) \mid \bar{p}x = \alpha,\ z = \bar{z}\}$

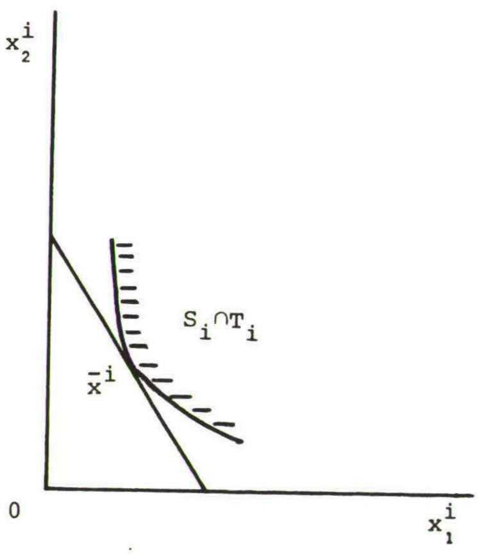

fig 2: $S_i \cap T_i =$
$\{(x^i, t_i) \mid x^i \in D_n, g_i(x^i) \geq \bar{t}_i\}$

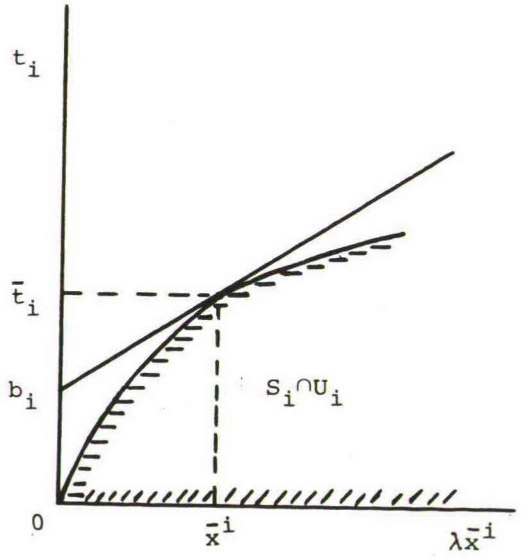

fig 3: $S_i \cap U_i =$
$\{(\lambda, t_i) \mid x^i = \lambda \bar{x}^i,$
$0 \leq t_i \leq g_i(x^i)\}$

The sets B and C have $(\bar{x},\bar{z})$ in common, so certainly $(\bar{x},\bar{z}) \in V$.
Let $V^1 = V \cap A$ i.e. $V^1 = \{(x,z) \mid p\ x + q\ \bar{z} = \beta\}$.
V and V^1 are invariable for scalar multiplication of p, q and β, so V^1 is always reducible to $V^1 = \{(x,z) \mid px = \alpha\}$.
There is left to prove that $p = \bar{p}$ i.e. $B = V^1$. B and V^1 both being $(n+m-1)$-dimensional hyperplanes in R^{n+m}, it is sufficient to prove: $B \subset V^1$.
Suppose $(x,z) \in B$ i.e. $\bar{p}\ x = \alpha,\ z = \bar{z}$.

Let $x = \bar{x} + x_r$. Hence $\bar{p}\ x = \bar{p}\ \bar{x} + \bar{p}\ x_r \Rightarrow \alpha = \alpha + \bar{p}\ x_r \Rightarrow$

$$\Rightarrow \bar{p}\ x_r = 0.$$

It holds that $(y,z) = (\bar{x}-x_r,\ \bar{z}) \in B$, for $\bar{p}\ y = \bar{p}\ \bar{x} - \bar{p}\ x_r = \alpha$

$$\left.\begin{array}{l} \text{If } (x,z) \in B \text{ then } px \le \alpha \Rightarrow p\bar{x}+px_r \le \alpha \Rightarrow px_r \le 0 \\ \quad (y,z) \in B \text{ then } py \le \alpha \Rightarrow p\bar{x}-px_r \le \alpha \Rightarrow px_r \ge 0 \end{array}\right\} \Rightarrow px_r = 0$$

Hence $px = py = \alpha$.
So $(x,z) \in B$ implies $(x,z) \in V^1$. This completes the proof.

3.3 The construction of the taxationvector at $\bar{\bar{L}}(u)$

Since $\bar{\bar{L}}(u)$ is a convex closed set in D_{n+m} (See 2.6), now we can state that given $\bar{p} \in D_n$ and $\bar{z} \in D_m$ there exist an element $(\bar{x},\bar{z}) \in \bar{\bar{L}}(u)$, a vector $q \in D_m$ and a scalar β in such a way that:

$$\bar{p}\ \bar{x} + q\ \bar{z} = \beta$$

$$\bar{p}\ \bar{x} + q\ z \geq \beta$$

It will be shown how to construct q and β. For all i $(i = 1...m)$ $f_i(u) - g_i(x^i) \leq \bar{z}_i$ should hold. For every u $f_i(u)$ is uniquely determined, hence $\bar{t}_i = f_i(u) - \bar{z}_i$ is uniquely determined, so $g_i(x^i) \geq \bar{t}_i$ should hold.

Consider $S_i = \{(x^i, t_i) \mid x^i \in D_n, 0 \leq t_i \leq g_i(x^i)\}$.
Since function $g_i(.)$ is concave, S_i is convex. Let $T_i = \{(x^i, t_i) \mid x^i \in D_n, t_i = \bar{t}_i\}$. Also T_i is convex.
Hence $S_i \cap T_i = \{(x^i, t_i) \mid x^i \in D_n, g_i(x^i) \geq \bar{t}_i\}$ is convex (fig.2).
So given $\bar{p} \in D_n$ there exist a point $(\bar{x}^i, \bar{t}_i) \in$ Bnd $(S_i \cap T_i)$ and a scalar α_i in such a way that:

$$\bar{p}\,\bar{x}^i = \alpha_i$$

and $\quad \bar{p}\,x^i \geq \alpha_i \quad$ for $\quad (x^i, t_i) \in S_i \cap T_i$.

Construct $U_i = \{(x^i, t_i) \mid \exists_{\lambda \geq 0} : x^i = \lambda \bar{x}^i, t_i \geq 0\}$.
U_i is convex.
Hence $S_i \cap U_i = \{(x^i, t_i) \mid \exists_{\lambda \geq 0} : x^i = \lambda \bar{x}^i, 0 \leq t_i \leq g_i(x^i)\}$ is convex.
In two dimensions one can reformalate $S_i \cap U_i$ as:

$$S_i \cap U_i = \{(\lambda, t_i) \mid x^i = \lambda \bar{x}^i, 0 \leq t_i \leq g_i(x^i)\} \quad \text{(Fig.3)}$$

Since $S_i \cap U_i$ is convex, a scalar b_i exists in such a way that $t_i = (\bar{t}_i - b_i)\lambda + b_i$ is a tangent line of $S_i \cap U_i$ in $(\bar{x}^i, \bar{t}_i)$ and $t_i \leq (\bar{t}_i - b_i)\lambda + b_i$ for $(x^i, t_i) \in S_i \cap U_i$.

We search for the supporting hyperplane $(\bar{p}x^i + q_i^1 t_i = \beta_i)$, spanned by the hyperplane $\bar{p}x^i = \alpha_i$ and the line $t_i = (\bar{t}_i - b_i)\lambda + b_i$.

$$\left.\begin{array}{ll} (\bar{x}^1, t_i) \in \text{hyperplane, so } \bar{p}\bar{x}^i + q_i^1 \bar{t}_i = \beta_i \Rightarrow \alpha_i + q_i^1 \bar{t}_i = \beta_i \\ (0, b_i) \in \text{hyperplane, so } \qquad q_i^1 b_i = \beta_i \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad q_i^1 = \frac{\alpha_i}{b_i - \bar{t}_i} \quad \text{and} \quad \beta_i = \frac{\alpha_i b_i}{b_i - \bar{t}_i}$$

Moreover the convexity of $L(u)$ implies, given $\bar{p}$, the existence of a point $\bar{x}^0$ and a scalar α_o in such a way that $\bar{p}\,\bar{x}^0 = \alpha_o$ and $\bar{p}\,x^0 \geq \alpha_o$ for $x^0 \in L(u)$.
The final equation of the supporting hyperplane of $\bar{\bar{L}}(u)$ at $(\bar{x},\bar{z})$ can be deduced as follows:

For all i ($i \neq 0$) holds: $\bar{p}\,x^i + \dfrac{\alpha_i}{b_i-\bar{t}_i}\,t_i = \dfrac{\alpha_i b_i}{b_i-\bar{t}_i}$

For $i = 0$ holds: $\bar{p}\,x^0 = \alpha_o$

Since $\sum_{i=0}^{m} x_i = x$ and $t_i = f_i(u) - z_i$ c.q. $t = F(u) - z$

one can state:

$$\bar{p}\,x + q^1(F(u) - z] = \alpha_o + \sum_{i=1}^{m} \frac{\alpha_i b_i}{b_i-f_i(u) + \bar{z}_i}$$

Let $q = -q^1$; hence

$$\bar{p}\,x + q\,z = \alpha_o + \sum_{i=1}^{m} \frac{\alpha_i}{b_i-f_i(u) + \bar{z}_i}[b_i-f_i(u)]$$

$$\text{with } q = \left[\frac{\alpha_1}{f_1(u)-\bar{z}_1-b_1}, \frac{\alpha_2}{f_2(u)-\bar{z}_2-b_2} \;--\; \frac{\alpha_m}{f_m(u)-\bar{z}_m-b_m}\right]$$

(3.3.1) Remark:

There may be circumstances that the concavity of the e.d.e.- function can be restricted to a subset of D_n. Then the reasoning is not carried out, based on $S_i = \{(x^i,t_i) \mid x^i \in D_n,\ 0 \leq t_i \leq g_i(x^i)\}$, but based on the convex hull of S_i. If the hyperplane to be constructed is supporting this convex hull at a point also belonging to S_i, the taxationvector is suitable to restrict the external diseconomies to the fixed maximum. See fig.4.

(3.3.2) Remark:

The existence of a taxationvector does not necessarily imply its uniqueness. In fig. 5 you see a situation of a infinite number of supporting hyperplane of $S_i \cap T_i$

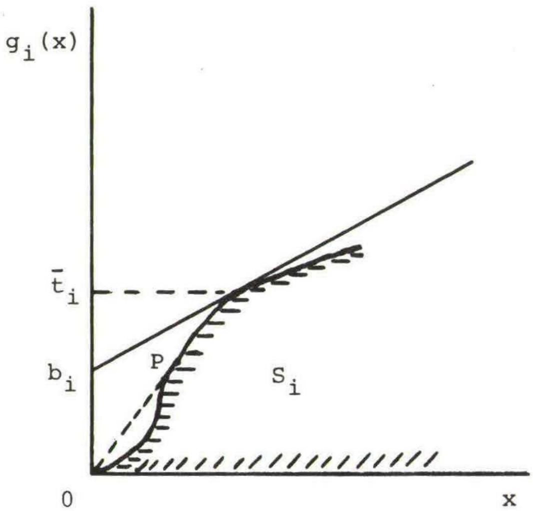

fig 4: convex hull of S_i

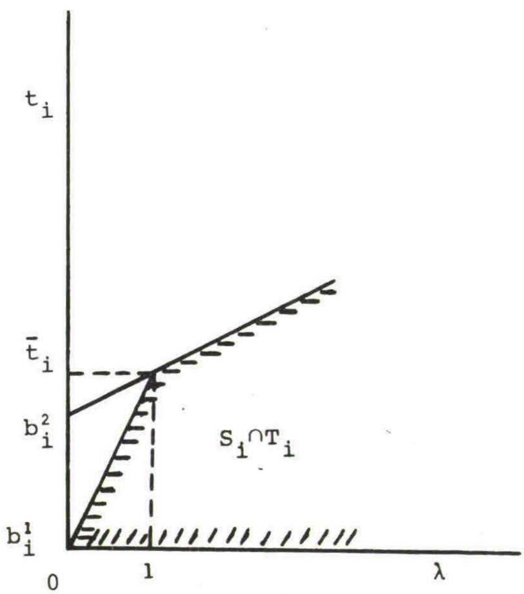

fig 5: more tangentlines at $S_i \cap T_i$ in $(\bar{x}^i, \bar{t}_i)$

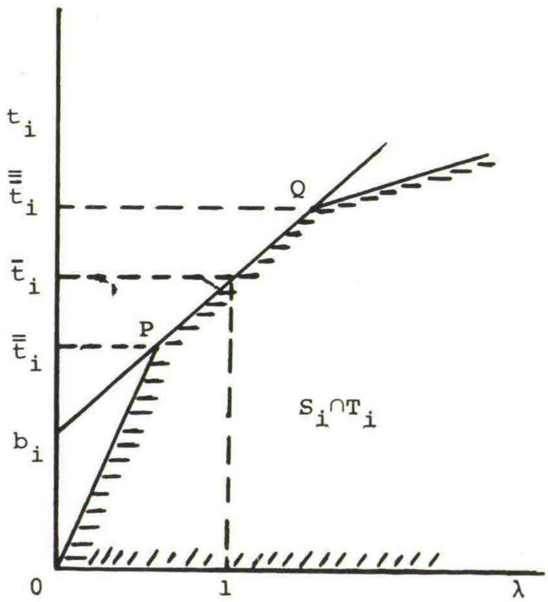

fig 6: $S_i \cap T_i$ not strictly convex in $(\bar{x}^i, \bar{t}^i)$

at $(\bar{x}^i,\bar{t}_i)$. Evidently a closed interval $[b_i^1,b_i^2] \subset [0,\bar{t}_i]$ can be found to quarantee for all $b_i \in [b_i^1,b_i^2]$ the existence of a suitable taxation. We remind you that we were looking for a minimal taxation, suited to restrict the external diseconomies. Clearly the choise of b_i^1 as smallest in the interval $[b_i^1,b_i^2]$ implies a minimal value of the taxation q_i.

(3.3.3) Remark:

The set $S_i \cap T_i$ may not be strict convex in $(\bar{x}^i,\bar{t}_i)$ i.e. $(\bar{x}^i,\bar{t}_i)$ is not an extreme point of $S_i \cap T_i$. Fig. 6. The points on the linesegment PQ are indifferent for the costminimizing producer. So only an elimination $\bar{\bar{t}}_i < \bar{t}_i$ may occur. If $(\bar{x}^i,\bar{t}_i)$ coincides with P, b_i can be maintained for the construction of the taxation, but for all other positions of $(\bar{x}^i,\bar{t}_i)$ on the linesegment PQ a very small increase of b_i to $b_i + \varepsilon$ $(\varepsilon>0)$ will enlarge the elimination of the external diseconomie to $\bar{\bar{t}}_i \geq \bar{t}_i$. In such a situation a minimal taxation can't be found, merely its infimum or greatest lower bound.

3.4 The properties of the taxationvector.

The relation existing between the level of the minimal resp. infimal taxation and on the other hand the price-vector of the "common" factors, the outputrate of the productionprocess and the upperlimit of the external diseconomy, can easily be deduced from the method of construction in the preceeding paragraph. Since according to the e.d.g.-function the outputrate and the established maximum are uniquely determining the necessary level of elimination, it is sufficient to consider the convex graph S_i of the e.d.e.-function to study the taxationproblem, like we did in the preceeding paragraph. That constructionmethod shows that for each combination

of u, $\bar{z}$ and $\bar{p}$, provided $u \geq 0$, $\bar{z} \geqq 0$ and $\bar{p} \in D_n$, a minimal or infimal taxation q_i can be found uniquely.
Hence q_i is a function of each combination $(u,\bar{z},\bar{p})$ and a fortiori a function of each of the elements of this combination, both others fixed.
Evidently the specification of this function depends on the concrete formulation of the relevant e.d.g.- and e.d.e.-function. Nevertheless some properties of this function can be stated.

Provided $u \geq 0$ and $\bar{z} \geqq 0$, q is a function of $p \in D$ satisfying:

p.1 If $p = 0$ $\Rightarrow$ For $\forall_i$: $f_i(u) \leq \bar{z}_i$ $q_i = 0$ (minimum)

For $\forall_i$: $f_i(u) > \bar{z}_i$ $q_i = 0$ (infimum)

p.2 If $p > 0$ $\Rightarrow$ For $\forall_i$: $f_i(u) \leq \bar{z}_i$ $q_i = 0$ (minimum)

For $\forall_i$: $f_i(u) > \bar{z}_i$ $q_i > 0$ (min. or inf.)

This is necessarily implied by the properties (f.4) and (f.5) of the e.d.e.-function.

p.3 If $p \geq 0$ $\Rightarrow$ For $\forall_i$: $f_i(u) \leq \bar{z}_i$ $q_i = 0$ (minimum)

For $\forall_i$: $f_i(u) > \bar{z}_i$ and $p_j = 0$ for every essential productionfactor x_j, then $q_i = 0$ (inf.)

For $\forall_i$: $f_i(u) > \bar{z}_i$ and an essential productionfactor x_j exists with $p_j > 0$ then $q_i > 0$ (min. or inf.)

p.4 If $p_2 = \lambda p_1 \Rightarrow q_2 = \lambda q_1$. For all i (i = 1...m) holds

$p_1 \bar{x}^i = \alpha_i \Rightarrow (\lambda p_1)\ \bar{x}^i = \lambda \alpha_i \Rightarrow q_2 = \lambda q_1$

since both b_i and $\bar{t}_i$ remain unchanged.

p.5 If $p \to + \infty \Rightarrow q \to + \infty$. For all i (i = 1...m) holds:

$p \to \infty \Rightarrow \alpha_i \to \infty$; $(\bar{t}_i - b_i)$ is always finite, hence $q \to \infty$.

If $p_j \to + \infty$ for x_j^i is essential $\Rightarrow q_i \to + \infty$. The proof is analogous.

p.6 If p is finite, then q is finite. For all i (i = 1...m) holds that, for finite u, α_i is finite, $(\bar{t}_i - b_i)$ is always finite, so q is finite.

Provided $p \in D_n$ and $\bar{z} \geqq 0$, q is a function of $u \geq 0$ satisfying.

u.1 If $u = 0 \Rightarrow q = 0$ (minimum)

u.2 If $u > 0 \Rightarrow$ For $\forall_i$: $f_i(u) \leq \bar{z}_i$ $q_i = 0$ (minimum)

For $\forall_i$: $f_i(u) > \bar{z}_i$ and an essential productionfactor x_j exists with $p_j > 0$ then $q_i > 0$ (min. or inf.).

For $\forall_i$: $f_i(u) > \bar{z}_i$ and $p_j = 0$ for every essential productionfactor x_j, then $q_i = 0$ (inf.)

u.3 If $u \to + \infty \Rightarrow$ If $p_j = 0$ for every essential productionfactor x_j then $q_i = 0$ (inf.). If an essential productionfactor x_j exists with $p_j > 0$ then $q_i \to + \infty$.

u.4 If u is finite, q is finite too. $f_i(u)$ is finite. (See f.4), hence according to (g.2) α_i is finite too and so q_i is finite. This reasoning holds for all i (i = 1...m).

Provided $p \in D_n$ and $u \geq 0$, q is a function of $\bar{z} \geqq 0$ satisfying:

z.1 If $\bar{z}_i = 0 \Rightarrow$ If $u = 0$ then $q_i = 0$ (minimum).

If $u > 0$ and an essential factor x_j exists with $p_j > 0$ then $q_i > 0$ (min. or inf.), since $\forall_i: f_i(u) > 0$ (See f.2).

If $u > 0$ and $p_j = 0$ for every essential factor x_j, then $q_i = 0$ (inf.).

z.2 If $\bar{z}_i > 0 \Rightarrow$ If $f_i(u) \leq \bar{z}_i$ then $q_i = 0$ (minimum).

If $f_i(u) > \bar{z}_i$ and an essential factor x_j exists with $p_j > 0$ then $q_i > 0$ (min. or inf.).

If $f_i(u) > \bar{z}_i$ and $p_j = 0$ for every essential factor x_j, then $q_i = 0$ (inf.).

z.3 If $\bar{z}_i \to +\infty \Rightarrow q_i \to 0$. This is necessarily implied by z.2.

z.4 If $\bar{z}_i$ is finite, q_i is finite too. This is necessarily implied by z.2 and (g.2).

(3.4.1) Remark:

If the e.d.e.-function is positively homogeneous of degree one i.e. $\forall_{\lambda \geq 0}\ g_i(\lambda x^i) = \lambda g_i(x^i)$, the set $S_i = \{(x^i, t_i) \mid x^i \in D_n,\ 0 \leq t_i \leq g_i(x^i)\}$ is a convex cone, Hence the intersection $S_i \cap T_i$ is a convex cone, spanned by the vectors $(\bar{x}^i, 0)$ and $(\bar{x}^i, \bar{t}_i)$. See fig. 7. The slope γ depends on p.

Clearly $b_i = 0$, hence $q_i = \frac{\alpha_i}{\bar{t}_i}$.

Provided p, q_i is identical for all $\bar{\bar{t}}_i > 0$. Choose arbitrarely $\bar{\bar{t}}_i > 0$. Let $\bar{\bar{t}}_i = \mu\bar{t}_i(\mu>0)$. According to the homogenity of the e.d.e.-function one can state $\bar{\bar{x}}^i = \mu\bar{x}^i$ and $\bar{\bar{\alpha}}_i = \mu\alpha_i$.

Hence $q_i = \dfrac{\bar{\bar{\alpha}}_i}{\bar{\bar{t}}_i} = \dfrac{\alpha_i}{\bar{t}_i}$.

This value of q_i is the infimal taxation. Establishing the taxation on $q_i + \varepsilon(\varepsilon>0)$ implies the complete elimination $\{f_i(u)\}$ of the external diseconomy.

3.5 The taxationvector at $\bar{\bar{L}}_1(u)$.

Since $\bar{\bar{L}}_1(u)$ is a convex closed set in D_{n+m} (See 2.11), we can state too, that provided $\bar{p} \in D_n$ and $\bar{z} \in D_m$ there exist an element $(\bar{x},\bar{z}) \in \bar{\bar{L}}_1(u)$, a vector $q \in D_m$ and a scalar β in such a way that:

$$\bar{p}\ \bar{x} + q\ \bar{z} = \beta$$

$$\bar{p}\ x + q\ z \geq \beta \quad \text{for} \quad (x,z) \in \bar{\bar{L}}_1(u)$$

Hence there is no doubt that for the productionstructure $\bar{\bar{L}}_1(u)$ too a minimal resp. infimal taxationvector can be found to restrict the level of external diseconomies.
But unfortunately, in this alternative situation a similar constructionmethod as in paragraph 3.3 is not available. There we could study separately the pricesystems, belonging to the technology of the desired commodity, L(u), and the distinct eliminationprocesses, and by summation integrate them in the final dual structure.
But now it is not impossible, that a inputvector x^0, yielding minimal costs $\bar{p}\ x^0 = \alpha_o$ for some pricevector $\bar{p}$ with respect to L(u), causes a high level of external diseconomies with high elimination costs β_o, while a suboptimal inputvector $\bar{x}^0$ with costs $\bar{p}\ \bar{x}^0 > \bar{p}\ \bar{x}^0 > \alpha_o$,

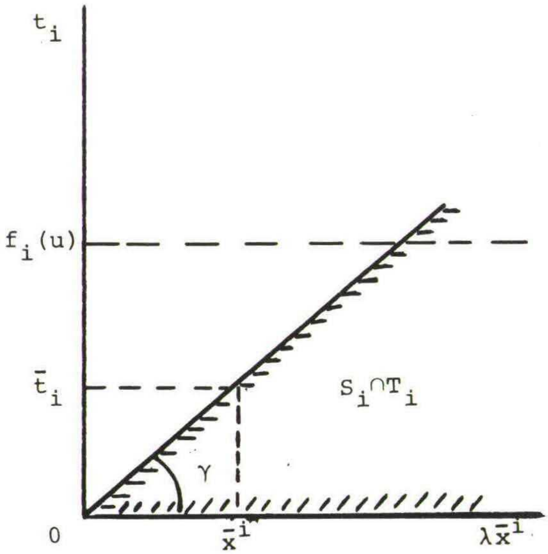

fig 7: $S_i \cap T_i$ for a positively linear homogeneous e.d.e.-function

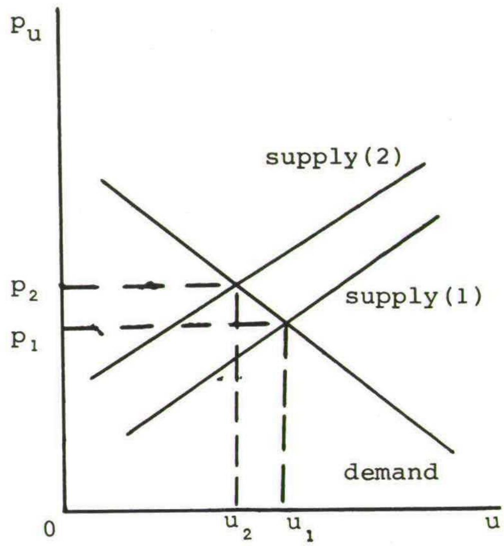

fig 9: The situation of elastic demand

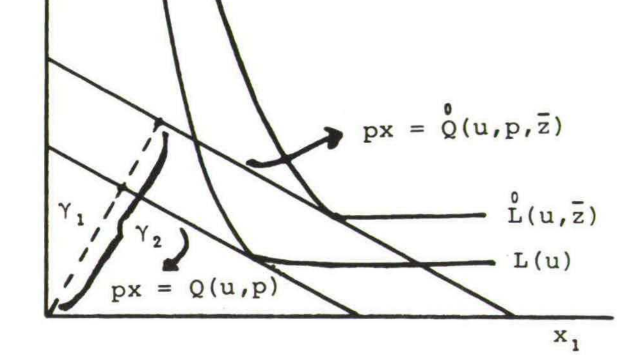

fig 8: The confrontation of L(u) and $\overset{0}{L}(u,\bar{z})$

can cause a relatively low level of external diseconomies with eliminationcosts lower than β_o. In the latter situation total costs may be lower.
Although it is possible to find a minimal costprice $\bar{p}\ \bar{x} + q\ \bar{z} = \beta$, a constructionmethod based on a separate treatment of the productiontechnology L(u) and the several eliminating processes cannot be applied here.

3.6 The interpretation of the model (micro-economically)

To avoid situations, mentioned in remarks (3.3.2) and (3.3.3), we suppose strict convexity of the production-structure. Moreover we restrict the story to $\bar{\bar{L}}(u)$, since all remarks hold for $\bar{\bar{L}}_1(u)$ too, but according to the preceeding paragraph in a more complex manner.
If the taxationvector q is established as $\bar{z}$ not to be exceeded, the producer is facing the following costs:

$$\bar{p}\ \bar{x} + q\ \bar{z} = \bar{p}\ \bar{x}^0 + \bar{p} \sum_{i=1}^{m} \bar{x}^i + q\ \bar{z}$$

$\bar{p}\ \bar{x}^0$: the minimal costs of producing outputrate u of the desired commodity.

$\bar{p} \sum_{i=1}^{m} \bar{x}^i$: minimal costs of eliminating the external diseconomies to $\bar{z}$.

$q\ \bar{z}$: additional charge for the level of external diseconomies.

Now we compare the situations before and after the introduction of the taxationvector q.

before

factorpayments	$\bar{p}\,\bar{x}^0$	sales	$p^1 u$

$$p^1 = \frac{\bar{p}\,\bar{x}^0}{u}$$

after (Case I)

factorpayments	$\bar{p}\,\bar{x}$	sales	$p^2 u$
tax	$q\,\bar{z}$	taxrepayment	$q\bar{z}$

$$p^2 = \frac{\bar{p}\,\bar{x}}{u}$$

after (Case II)

factorpayments	$\bar{p}\,\bar{x}$	sales	$p^3 u$
tax	$q\,\bar{z}$		

$$p^3 = \frac{\bar{p}\,\bar{x} + q\,\bar{z}}{u}$$

We assume these confrontations to balance, e.g. due to competitive market conditions. For the moment we also assume the outputrate being fixed on the level u; i.e. demand is inelastic.

Before the introduction of the taxation the production-costs amount to $\bar{p}\,\bar{x}^0$, while by setting the sellingprice to p^1 the turnover just equals the costs.

The situation afterwards can be considered in two different ways.

First one can say that - provided a maximum level of external diseconomies established and not exceeded by the producer - it is unreasonable to charge him with an additional amount $q\,\bar{z}$. This amount should be repaid by the taxreceiver (Case I). Then the sellingprice p^2 yields a turnover equal to the production- and eliminationcosts. The amount $q\,\bar{z}$ may however be considered as compensation for causing external diseconomies; it is true the maximum level is not exceeded, but nevertheless relatively scarce means are used and that should be paid for.

In the latter case (II) the sellingprice p^3 should be higher to cover this compensation too.
In this case the taxation q is not only an instrument to avoid too much pollution etc., but also an instrument to fill the public treasury.

Assuming the taxreceiver to repay the amount $q\,\bar{z}$ is equivalent with assuming the producer to face the productionstructure $\overset{0}{L}(u,\bar{z})$. By confronting the productionstructures $L(u)$ and $\overset{0}{L}(u,\bar{z})$ the increase of costs due to the obliged elimination of external diseconomies can be shown by a simple geometric relation (fig. 8). For $L(u)$ as well as for $\overset{0}{L}(u,\bar{z})$ a factor minimal costfunction exists satisfying:

$$Q(u,p) = \| \gamma_1 \| \, . \, \| p \| \text{ for } p \neq 0$$

$$\overset{0}{Q}(u,p,\bar{z}) = \| \gamma_2 \| \, . \, \| p \| \text{ for } p \neq 0 \text{ See ([4], page 81)}$$

Hence the increase of costs, expressed in orginal costs, equals:

$$\frac{\overset{0}{Q}(u,p,\bar{z}) - Q(u,p)}{Q(u,p)} = \frac{\| \gamma_2 \|}{\| \gamma_1 \|} - 1$$

Since $\overset{0}{L}(u,\bar{z}) \subset L(u)$, $\| \gamma_2 \| \geq \| \gamma_1 \|$ is true.

Note that not only the concrete specification of $L(u)$ and $\overset{0}{L}(u,\bar{z})$, but also the pricevector $\bar{p} \in D_n$ influence the relation between $\| \gamma_2 \|$ and $\| \gamma_1 \|$.

The assumption of inelastic demand is clearly very unrealistic. The preceeding argument is still valid, if more realistic assumptions on demand are established. Assume the existence of a normal i.e. decreasing demandfunction. On $L(u)$ a factor minimal costfunction $Q(u,p)$ is defined. A supply-function for u is easily deduced

by assigning to each $u \geq 0$ the value of $Q(u,p)/u$.
Under conditions of non-increasing returns to scale the supply-function is non-decreasing. See ([4] , page 83, Q12).
The intersection of these functions determines the output-rate u_1 and the price p_1 (fig. 9).
In (3.4) we showed, that, provided $\bar{z} \in D_m$ and $\bar{p} \in D_n$, q is a function of u. So for each $u \geq 0$ a relevant q can be found, hence a supply-function for u can be deduced from the factor minimal costfunction $Q(u,p,q)$ on $\bar{\bar{L}}(u)$ by assigning to each $u \geq 0$ the value of $Q(u,p,q)/u$.
The intersection of this function and the demandfunction determines the outputrate u_2, the price p_2 and moreover the taxationvector q. (fig. 9)
The confrontation of proceeds and expenditures have to be constructed now with respect to u_1 and u_2. The discussion about the repayment of $q\,\bar{z}$ remains unchanged.

Under assumption of inelastic demand more "common" productionfactors have to be assigned to the production-process than needed for the mere production of outputrate u. It is reasonable to suppose an upper-limit to the availability of the "common" productionfactors like capital and labour. Hence if all these factors were employed, the shift of a certain amount of the factors to our productionprocess necessarily reduces the aggregate output in society. This statement holds clearly also for situations of more elastic demand. In general one can state that a decrease of the maximum level of external diseconomies is associated with a decrease of aggregate output. Now the problem is, which combination of material output and external diseconomies is optimal, and optimal in which way. To give an answer to this question we interprete our model in a macro-economic sense.

3.7 The social utility function

The productiontechnologies $L(u)$, $\bar{L}(u)$, $\bar{\bar{L}}(u)$, $\overset{0}{L}(u,\bar{z})$, $\bar{L}_1(u)$,

$\bar{\bar{L}}_1(u)$ and $\overset{0}{L}_1(u,\bar{z})$ may be interpreted as blueprints of technical possibilities of a macro-system. The problem of aggregating the several outputs is ignored here; we are dealing with one aggregate output which level is denoted by u.

Consider Graph A = $\{(x,z,u) \mid u \geq 0,\ (x,z) \in \bar{\bar{L}}(u)\}$. The closedness of $\bar{\bar{L}}(u)$ implies the closeness of Graph A.

Let N = $\{(x,z,u) \mid u \geq 0,\ z \geqq 0,\ x = \bar{x}\}$. N is a closed set.

Consider M = Graph A $\cap$ N = $\{(x,z,u) \mid u \geq 0,\ x = \bar{x},\ (x,z) \in \bar{\bar{L}}(u)\}$.

M is the set of all those combinations of u and z that are feasible with respect to the limited available factors ($x = \bar{x}$). It will be shown that M is a compact set.

1) M is not empty. $(\bar{x},0,0) \in$ Graph A; $(\bar{x},0,0) \in N$; hence $(\bar{x},0,0) \in M$.

2) M is closed. Since Graph A and N are closed sets, their intersection is also closed.

3) M is bounded. On $\bar{\bar{L}}(u)$ a productionfunction $F(x,z)$ is defined. Choose arbitrarely $(x,z,u) \in M$. Hence $0 \leq u \leq F(\bar{x},z)$ i.e. $\Phi(x^0) \geq u$

$$\forall_i,\ i = 1\ldots m \qquad z_i \geq f_i(u) - g_i(x^i) \geq 0$$

$$\forall_i,\ i = 0\ldots m \qquad x^i \geqq 0$$

$$\sum_{i=0}^{m} x^i = \bar{x}$$

Since $x^0 \leqq \bar{x}$, x^0 is finite and hence $\Phi(x^0)$ is finite (A.2) en therefore u is bounded. Since $0 \leq z_i \leq f_i(u)$ and $f_i(u)$ is finite for finite u, hence z_i is bounded. So M is bounded. Hence M is a compact set.

It is worth mentioning that the compactness of M has nothing to do with the concavity of the e.d.g.-function. Substituting $\bar{\bar{L}}_1(u)$ for $\bar{\bar{L}}(u)$ doesn't alter the reasoning and an alternative compact set can be found.

The convexity of the alternative e.d.g.-function nor the concavity of the e.d.e.-function are necessary.

Now that we have found the feasible set M, we have to choose an optimum in it. So we need an objectfunction to maximize.
We assume the existence of a social utility function. According to the Weierstrass's Theorem the condition of continuity of the social utility function is sufficient to find at least once a maximum over the set M. Moreover this maximum is a boundary point of the set M, if some condition of monotonicity is satisfied i.e. the combination $(u_1 z_1)$ is at least as preferable as the combination $(u_2 z_2)$ with $u_2 \leq u_1$ and $z_2 \geqq z_1$.
Now that we have found a social optimum $(\bar{u},\bar{z})$ for $x = \bar{x}$, we wish to inquire if this optimum is sustained by a pricesystem with respect to $\bar{\bar{L}}(u)$. Clearly the point $(\bar{x},\bar{z})$ is a boundary point of the set $\bar{\bar{L}}(\bar{u})$. The convexity of $\bar{\bar{L}}(\bar{u})$ implies the existence of a pricevector (p,q) and a scalar β in such a way that:

$$p\,\bar{x} + q\,\bar{z} = \beta$$

$$\text{and } p\,x + q\,z \geq \beta \quad \text{for} \quad (x,z) \in \bar{\bar{L}}(\bar{u})$$

So if this price- and taxation system is established, the cost minimizing behaviour of the producers quarantees the attainment of the social optimum.

(3.7.1) <u>Remark</u>:

The social utility function may attain a maximum more than once over the set M. Hence the social optimum is not necessarily unique. Therefore the sustaining price- and taxation system is not unique too.

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